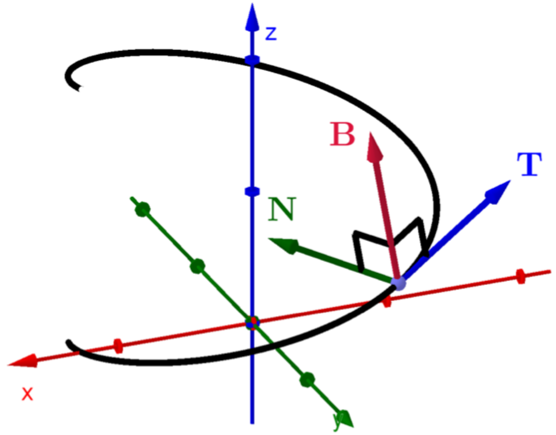
2.4 Frenet frame



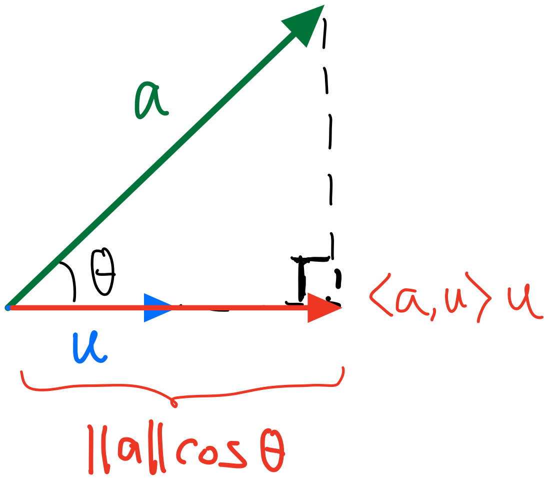
Definition 2.4.1 (Binormal). *Let* $\mathbf{r}(t)$ *be a space curve with curvature* $\kappa(t) > 0$ *for any* t. *We define the unit* **binormal** *to the curve by*

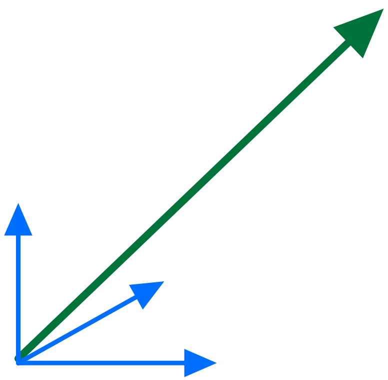
$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).$$

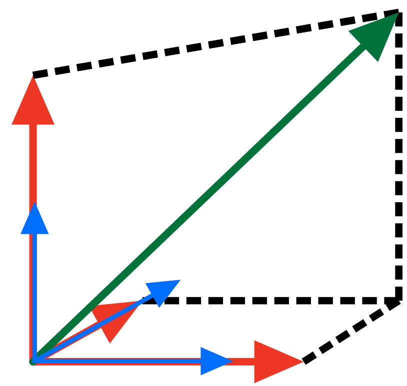
T, N, B are pairwisely orthogonal unit vectors (called orthonormal basis)

Observation: If $u, v, w \in \mathbb{R}^3$ are orthonormal, then for any $a \in \mathbb{R}^3$,

$$a = \langle a, u\rangle u + \langle a, v\rangle v + \langle a, w\rangle w$$







$$\langle a, u\rangle = \|a\|\,\|u\| \cos\theta = \|a\| \cos\theta$$

Q Let $\vec{r}(s)$ be parametrized by arclength. Express T', N', B' in terms of T, N, B

- $T' = \kappa N = 0T + \kappa N + 0B$

$N = \dfrac{T'}{\|T'\|}$ $\kappa = \dfrac{\|T'\|}{\|r'\|} = \|T'\|$

- $N' = \underline{\langle N', T\rangle} T + \underline{\langle N', N\rangle} N + \underline{\langle N', B\rangle} B$

$\langle N, N\rangle \equiv 1 \Rightarrow \langle N, N\rangle' = 0$

$\Rightarrow 2\langle N, N'\rangle = 0 \Rightarrow \langle N', N\rangle = 0$

$\langle N, T'\rangle = \langle N, \kappa N\rangle$

$= \kappa\langle N, N\rangle = \kappa$

$\langle N, T\rangle \equiv 0 \Rightarrow \langle N, T\rangle' \equiv 0$

$\Rightarrow \langle N', T\rangle + \langle N, T'\rangle = 0$

$\Rightarrow \langle N', T\rangle = -\langle N, T'\rangle = -\langle N, \kappa N\rangle = -\kappa\langle N, N\rangle = -\kappa$

$\langle N', B\rangle = ?$ New definition:

Definition 2.4.2 (Torsion). *Let* $\mathbf{r}(t)$ *be a space curve with curvature* $\kappa(t) > 0$ *for any* t*. The* **torsion** *of the curve at* $\mathbf{r}(t)$ *is defined by*

$$\tau = \left\langle \frac{d\mathbf{N}}{ds}, \mathbf{B} \right\rangle$$

where s *is a arc length parameter, which means* $\dfrac{ds}{dt} = \|\mathbf{r}'(t)\|$*. Equivalently, we have*

$$\tau(t) = \left\langle \frac{\mathbf{N}'(t)}{\|\mathbf{r}'(t)\|}, \mathbf{B}(t) \right\rangle$$

Proposition 2.4.3. *Let* $\mathbf{r}(t)$ *be a space curve with curvature* $\kappa(t) > 0$ *for any* t. *Then*

$$\tau = \frac{\langle \mathbf{r}' \times \mathbf{r}'', \mathbf{r}''' \rangle}{\|\mathbf{r}' \times \mathbf{r}''\|^2}.$$

$$u \times u = 0$$

$$\begin{cases} r' = \|r'\| T \\ r'' = \dfrac{\langle r', r'' \rangle}{\|r'\|} T + \kappa \|r'\|^2 N \end{cases}$$

$$\frac{dT}{dt} = \kappa v N$$

$$r''' = \left[\frac{d}{dt}\left(\frac{\langle r', r'' \rangle}{\|r'\|}\right)\right] T + \frac{\langle r', r'' \rangle}{\|r'\|} \underbrace{\frac{dT}{dt}} + \left[\frac{d}{dt}\left(\kappa \|r'\|^2\right)\right] N + \kappa \|r'\|^2 \frac{dN}{dt}$$

$$\kappa v N$$

$$\cancel{\kappa N}$$

$$r' \times r'' = \kappa \|r'\|^3 B$$

$$\langle B, T \rangle = \langle B, N \rangle = 0$$

$$\Rightarrow \langle r' \times r'', r''' \rangle = \left\langle \kappa \|r'\|^3 B, \kappa \|r'\|^2 \frac{dN}{dt} \right\rangle = \kappa^2 \|r'\|^6 \left\langle B, \frac{\frac{dN}{dt}}{\|r'\|} \right\rangle = \|r' \times r''\|^2 \tau$$

Theorem 2.4.4 (Frenet formula). *Let* $\mathbf{r}(s)$ *be a regular space curve parametrized by arc length with curvature* $\kappa(s) > 0$ *for any* s. *Then*

$$\begin{cases} \mathbf{T}'(s) &= & \kappa\mathbf{N} & & ① \\ \mathbf{N}'(s) &= -\kappa\mathbf{T} & & +\tau\mathbf{B} & ② \\ \mathbf{B}'(s) &= & -\tau\mathbf{N} & & ③ \end{cases}$$

discussed

We may write the formula in matrix form

$$\frac{d}{ds}\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}$$

For ③

$$B' = \langle B', T\rangle T + \langle B', N\rangle N + \langle B', B\rangle B$$

$\langle B, B\rangle \equiv 1 \Rightarrow \langle B, B\rangle' \equiv 0 \quad 2\langle B', B\rangle = 0 \Rightarrow \langle B', B\rangle = 0$

$\langle B, T\rangle \equiv 0 \Rightarrow \langle B, T\rangle' \equiv 0 \quad \langle B', T\rangle + \langle B, T'\rangle = 0, \quad \langle B', T\rangle = -\langle B, \kappa N\rangle = 0$

$\langle B, N\rangle \equiv 0 \Rightarrow \langle B', N\rangle = -\langle B, N'\rangle = -\tau$

Definition 2.4.5 (Plane curve). *We say that a space curve* **r** *is a* **plane curve** *if there exists a unit vector* **n** *such that*

$$\langle \mathbf{r}, \mathbf{n} \rangle = a$$

is a constant.

$\vec{n} = (n_1, n_2, n_3)$ $\quad \vec{X} = (x, y, z)$

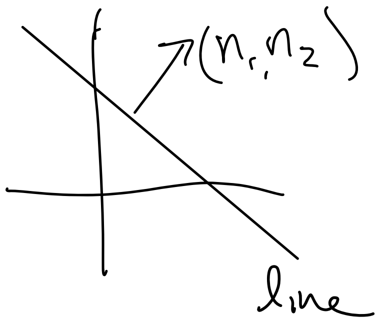
$\langle \vec{X}, \vec{n} \rangle = a$

$n_1 x + n_2 y + n_3 z = a$

equation of a plane

Analog in $\mathbb{R}^2$

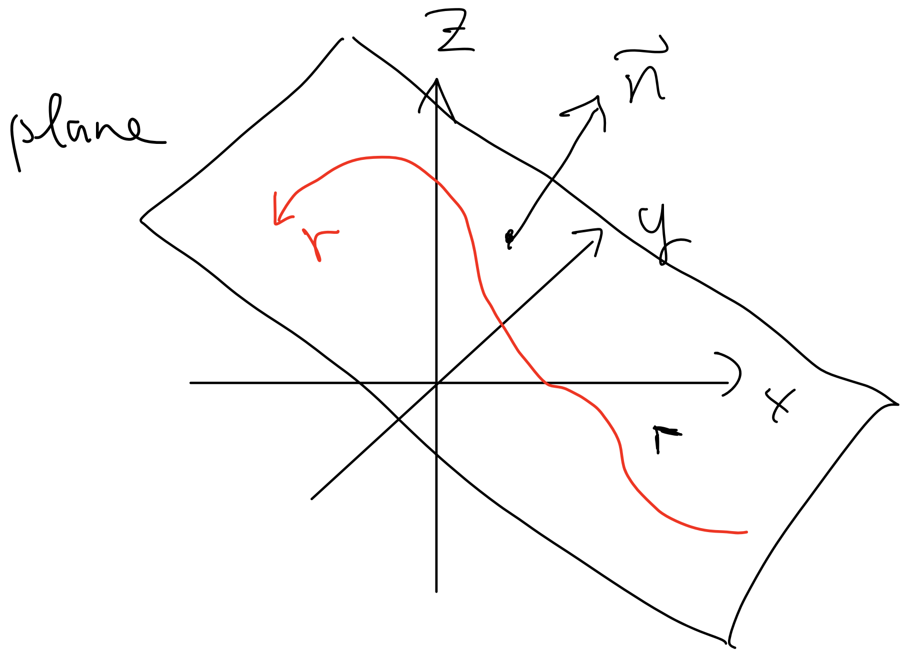
$n_1 x + n_2 y = a$



$\vec{r} = (r_1, r_2, r_3)$ satisfies

$\langle \vec{r}, \vec{n} \rangle = a$

$\Rightarrow$ $\vec{r}$ lies on the plane



Proposition 2.4.6. *Let* $\mathbf{r}(t)$ *be a regular parametrized space curve with curvature* $\kappa(t) > 0$ *for any* t. *Then* $\mathbf{r}$ *is a plane curve if and only if its torsion* $\tau(t) = 0$ *for any* t.

Plane curve $\Leftrightarrow$ $\tau(t) \equiv 0$

Pf $(\Rightarrow)$

$$\langle r, n \rangle = a \quad \forall s$$

$$\frac{d}{dt}: \quad \langle r', n \rangle = 0$$

$$\langle r'', n \rangle = 0$$

$$\langle r''', n \rangle = 0$$

Rmk $n \neq N$

$$\tau = \frac{\langle r' \times r'', r''' \rangle}{\|r' \times r''\|^2} = \frac{\langle \alpha \vec{n}, r''' \rangle}{\|r' \times r''\|^2} = 0$$

$r' \times r'' = \alpha \vec{n}$ for some α

$(\Leftarrow)$ Assume $\tau(t) \equiv 0$

$$B' = -\tau N = 0 \Rightarrow B \equiv \vec{B_0} \text{ constant}$$

$$\frac{d}{ds} \langle r, \vec{B_0} \rangle = \left\langle \frac{dr}{ds}, \vec{B_0} \right\rangle = \langle T, \vec{B_0} \rangle = \langle T, B \rangle = 0$$

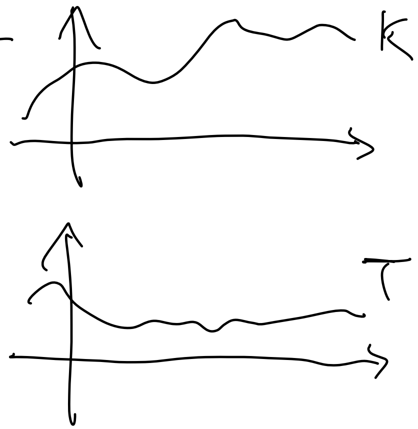
$\boxed{12:20}$

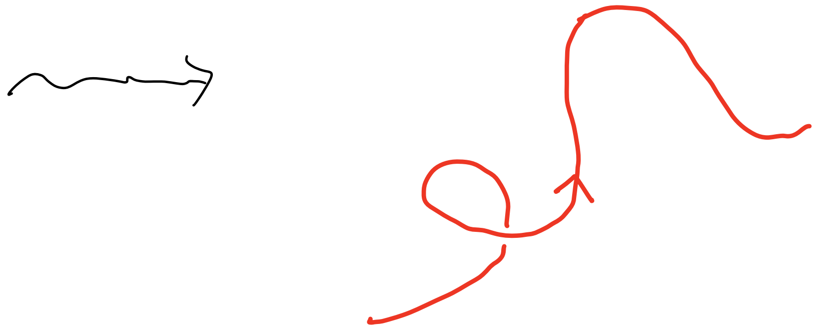
$\langle r, \vec{B_0} \rangle$ is constant

$\Rightarrow \langle r, B_0 \rangle = a$

Theorem 2.4.7 (Fundamental theorem of space curves)**.** *Let $\kappa(s), \tau(s) > 0$ be two positive functions. Then there exists unique, up to rigid transformation, space curve $\mathbf{r}(s)$ parametrized by arc length with curvature $\kappa(s)$ and torsion $\tau(s)$.*

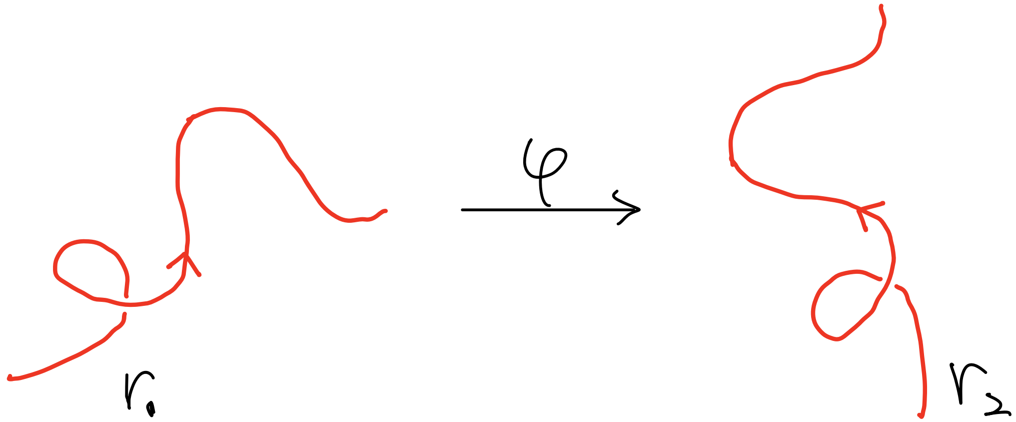
Existence: Any





$\exists r(s)$ in $\mathbb{R}^3$ with such κ, τ

Uniqueness If $r_1(s)$, $r_2(s)$ have same $\kappa(s)$, $\tau(s)$, then there is a rigid transformation (*) $\varphi: \mathbb{R}^3 \to \mathbb{R}^3$ s.t. $r_2(s) = \varphi(r_1(s))$



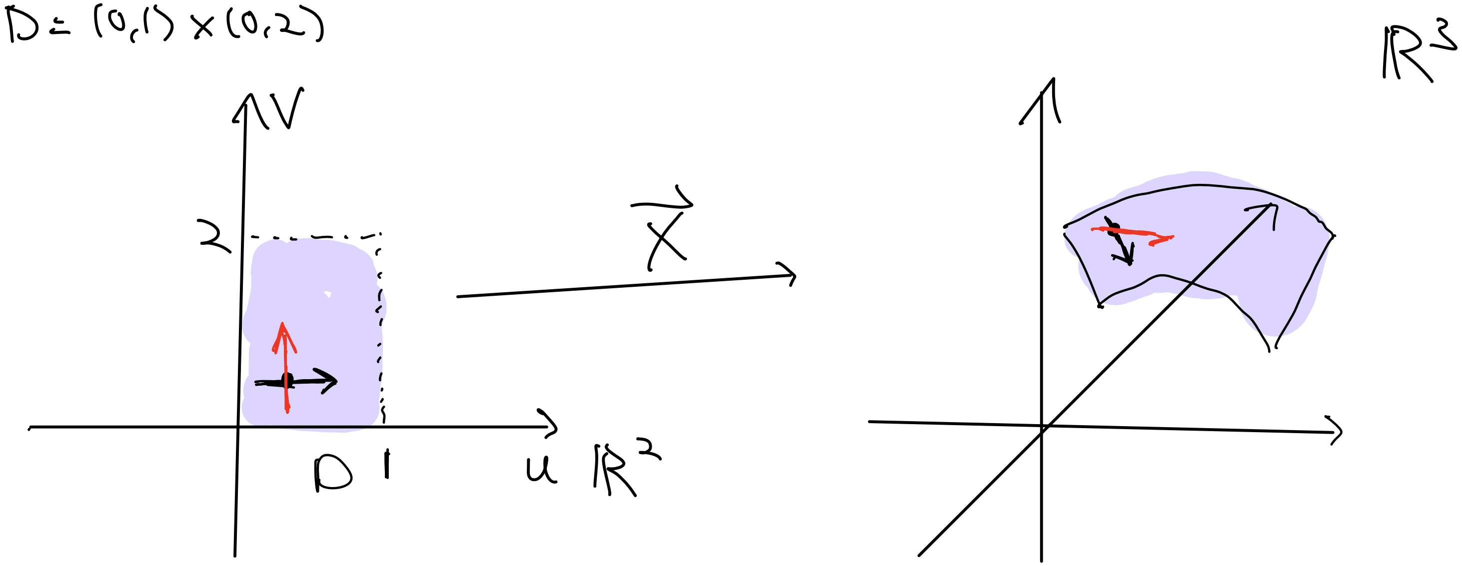
(*) means $\|\varphi(a) - \varphi(b)\| = \|a - b\|$ for any $a, b \in \mathbb{R}^3$ (Composition of rotation, reflection, translations)

3 Surfaces

3.1 Regular parametrized surfaces

Definition 3.1.1 (Regular parametrized surface)**.** *A* **regular parametrized surface** *is a differentiable function* $\mathbf{x} : D \to \mathbb{R}^3$*, where* $D \subset \mathbb{R}^2$ *is an open connected subset, such that* $\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$*, for any* $(u, v) \in D \subset \mathbb{R}^2$*. The image* $S = \mathbf{x}(D) \subset \mathbb{R}^3$ *is called a* **regular surface**.

eg $D = (0,1) \times (0,2)$



① For $\mathbf{x}(u, v)$, we denote $\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u}$ and $\mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v}$ to be the partial derivatives

② $X_u \times X_v \neq 0 \Rightarrow X_u, X_v$ are non-zero and not in the same/opposite directions

See next page

① <u>Partial derivatives</u> (derivative with respect to one of the variables)

$f(u,v) = u^2 v + v + 4u$ (two-variable functions)

$f_u = \frac{\partial f}{\partial u}(u,v) = 2uv + 0 + 4$ (Regard v as constant)

$f_v = \frac{\partial f}{\partial v}(u,v) = u^2 + 1 + 0 = u^2 + 1$

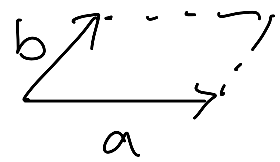
$f_{vv} = (f_v)_v = 0$

$f_{uu} = 2v$

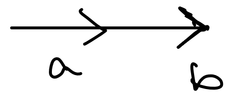
$f_{uv} = (f_u)_v = 2u$

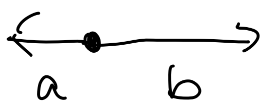
<u>Fact</u> $f_{vu} = f_{uv}$

② $a, b \in \mathbb{R}^3$

$\vec{a} \times \vec{b} \neq 0 \Rightarrow$ 

has non-zero area

$\vec{a} \times \vec{b} = 0$ if 



Definition 3.1.2 (Tangent space)**.** *Let S be a regular surface with parametrization $\mathbf{x}(u, v)$. The* **tangent space** *of S at $p = \mathbf{x}(u, v)$ is*

$$T_pS = \{\alpha \mathbf{x}_u + \beta \mathbf{x}_v : \alpha, \beta \in \mathbb{R}\} \subset \mathbb{R}^3.$$

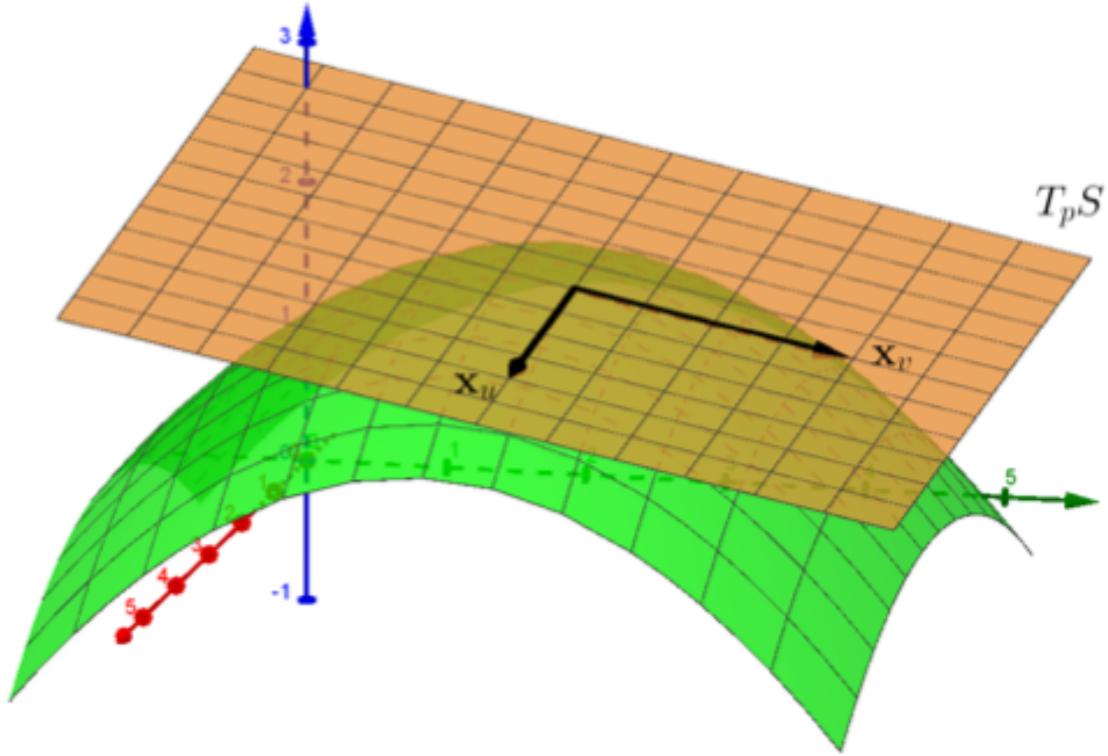
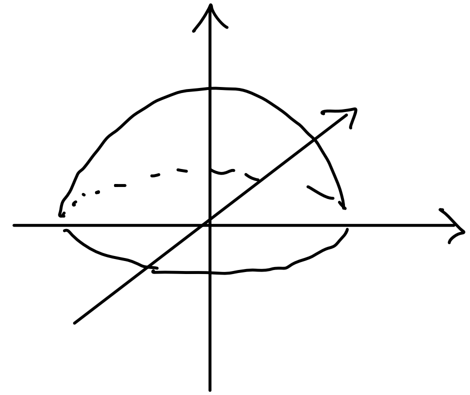


Figure 10: Tangent space

<u>Sphere</u> $\{(x,y,z) : x^2+y^2+z^2=r^2\}$ hollow 空心

Parametrization? $\vec{X}(x,y) = \left(x, y, \underbrace{\sqrt{r^2-x^2-y^2}}\right)$

$x^2+y^2<1$ $\geq 0 \Rightarrow$ only upper hemisphere



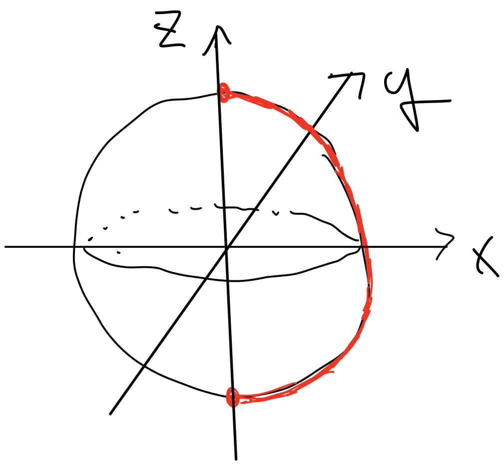
Better Parametrization $\vec{X}(\phi,\theta) = (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)$

$(\phi,\theta) \in (0,\pi) \times (0,2\pi)$

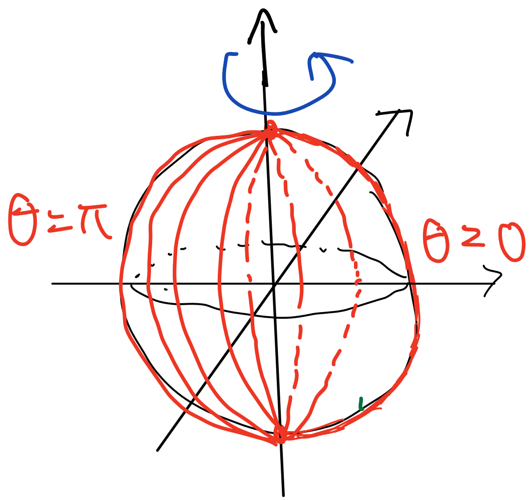
Take $\theta = 0$, $\phi \in (0,\pi)$

$\vec{X}(\phi,0) = (r\sin\phi, 0, r\cos\phi)$

$\theta \in (0,2\pi)$



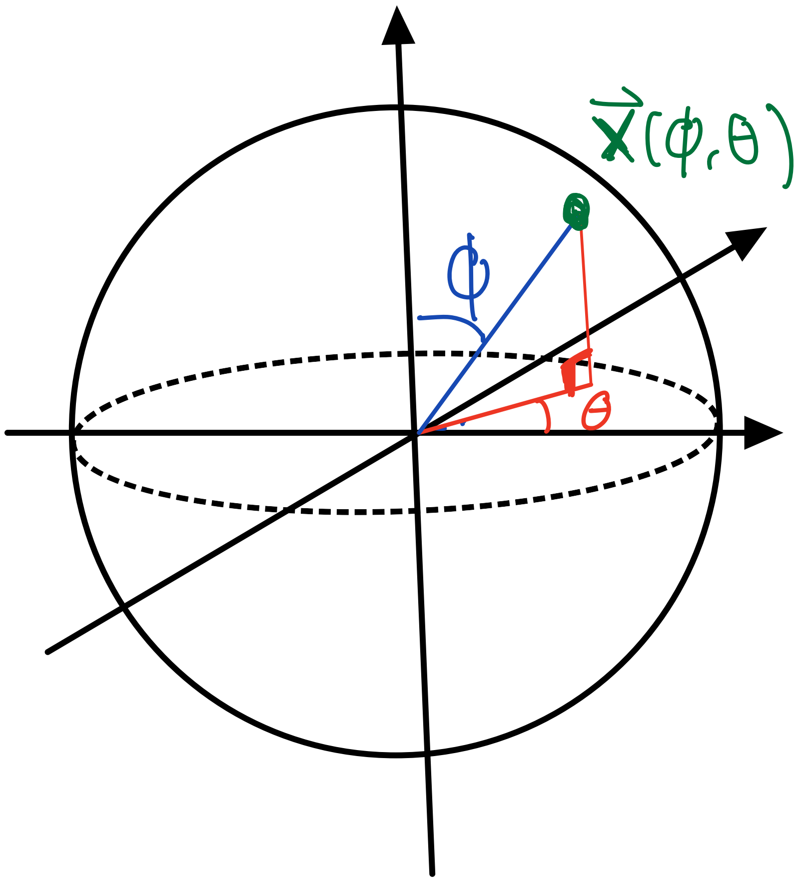
Rotate around z-axis $\longrightarrow$



1. *Sphere: Let $r > 0$ be a positive real number. The function*

$$\mathbf{x}(\phi, \theta) = (r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi), \ \textit{for } (\phi, \theta) \in (0, \pi) \times (0, 2\pi)$$

defines a **sphere** *of radius r centered at the origin.*



(r, θ, ϕ)

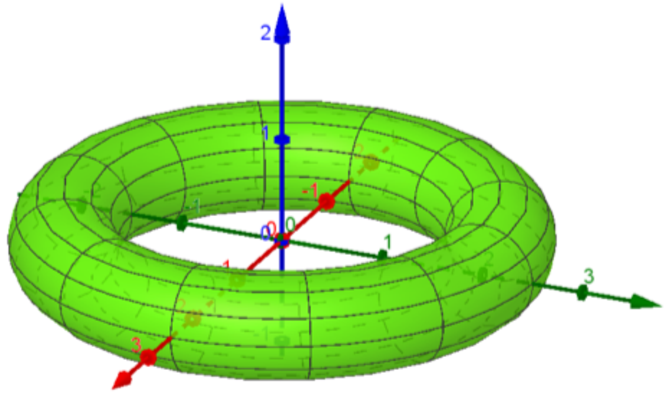
Spherical coordinate

2. *Torus: Let $R > r > 0$ be positive real numbers. The function*

$$\mathbf{x}(\phi, \theta) = ((R+r\sin\phi)\cos\theta, (R+r\sin\phi)\sin\theta, r\cos\phi), \quad \textit{for } \phi, \theta \in (0, 2\pi)$$

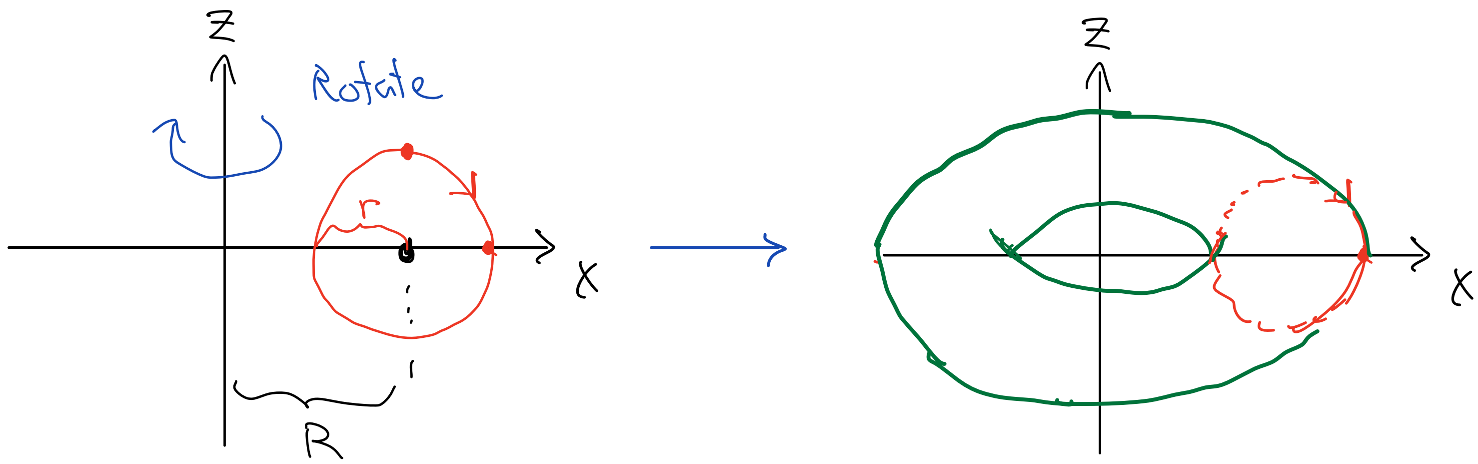
defines a regular surface which is called **torus**.

$R > r > 0$



If $\theta = 0$ $\quad X(\phi, 0) = (R + r\sin\phi, 0, r\cos\phi)$

$$= (R, 0, 0) + r(\sin\phi, 0, \cos\phi)$$



$\theta = 0$

$\phi \in (0, 2\pi)$

rotate around z-axis

$\theta \in (0, 2\pi)$

$\phi \in (0, 2\pi)$

3. *Helicoid: Let $a > 0$ be positive real numbers. The function*

$$\mathbf{x}(u, \theta) = (u\cos\theta, u\sin\theta, a\theta), \ \textit{for } u, \theta \in \mathbb{R}$$

defines a regular surface which is called **helicoid**.